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APPROXIMATION OF THE OPTIMAL COMPENSATOR

FOR A LARGE SPACE STRUCTURE

Michael K. Mackay

Paper presented at

WORKSHOP ON

APPLICATIONS OF DISTRIBUTED SYSTEM THEORY TO THE

CONTROL OF LARGE SPACE STRUCTURES

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This paper considers the approximation of the optimal compensator for a Large Space Structure. The compensator is based upon a solution to the Linear Stochastic Quadratic Regulator problem. Colocation of sensors and actuators is assumed. A small gain analytical solution for the optimal compensator is obtained for a single input/single output system, i.e., certain terms in the compensator can be neglected for sufficiently small gain. The compensator is calculated in terms of the kernal to a Volterra integral operator using a Neumann series. The calculation of the compensator is based upon the (CONT.)

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ITEM #20, CONTINUED: C_o semigroup for the infinite dimensional system. A finite dimensional approximation of the compensator is, therefore, obtained through analysis of the infinite dimensional compensator which is a compact operator.

APPROXIMATION OF THE OPTIMAL COMPENSATOR FOR A LARGE SPACE STRUCTURE[1,2]

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ABSTRACT

This paper considers the approximation of the optimal compensator for a Large Space Structure. The compensator is based upon a solution to the Linear Stochastic Quadratic Regulator problem. Colocation of sensors and actuators is assumed. A small gain analytical solution for the optimal compensator is obtained for a single input/single output system, i.e., certain terms in the compensator can be neglected for sufficiently small gain. The compensator is calculated in terms of the kernal to a Volterra integral operator using a Neumann series. The calculation of the compensator is based upon the C_0 semigroup for the infinite dimensional system. A finite dimensional approximation of the compensator is, therefore, obtained through analysis of the infinite dimensional compensator which is a compact operator.

C sub o

1.0 INTRODUCTION

One of the distinguishing properties of a Large Space Structure (LSS) is that it is a distributed parameter system and, hence, an infinite dimensional mathematical model is required for its description. In most applications, active control of shape, attitude, and structural vibrations will be necessary. For such problems, formulation of the control problem as a steady-state (infinite time) linear quadratic regulator is natural. The main advantage for considering the infinite time case is, of course, that the optimal control gain is time invariant. Implementation of the optimal feedback control will generally require an estimate of the system state. For the stochastic problem, the optimal state estimate is provided by an infinite dimensional Kalman filter. However, it is at this point where a singificant gap exists between theory and practice in that implementation of a infinite dimensional filter is not generally possible.

The most popular solution to this dilemma at the present time seems to be reduced order modeling of the system, see for example [1],[2],[3], and [4]. Usually, a modal representation of the system is assumed. The basic idea is then to evaluate various computable criteria representing the significance of various modes and then select some finite subset of modes to represent the dominant dynamics of the system. A linear quadratic regulator (controller/estimator combination) for this finite dimensional model is then designed using standard methods. The stability of the system using the resulting compensator is usually checked by computing the eigenvalues of the closed loop system using a high order "truth model" to represent the LSS dynamics, where the truth model is of finite order, but of much higher order than the compensator.

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2 Paper has been revised (8/6/82) from version distributed at workshop [illegible]

There are two basic problems with the above approach. The first is that the approximation is performed upon the system rather than the compensator. In effect, the truncation of an infinite number of modes is tantamount to "approximating" an unbounded, infinite dimensional operator by one that is bounded and finite dimensional. Thus, the infinite dimensionality of the problem is never used in designing the compensator. The second problem is that the stability of the closed loop system is also checked against a finite order model and the closed loop stability of the actual infinite dimensional system is inferred essentially on faith.

To avoid these pitfalls essentially requires that we have a partial differential equation (PDE) to represent the system. For complicated structures this may be asking too much. Hence, the method of reduced order modeling may be as viable an approach as any in those cases. However, we can, and should, study the problem of approximating the optimal compensator for appropriate systems where the PDE is known, since we may gain insight into the approximation problem that would apply to more general and more complicated systems.

This paper considers the compensator design and approximation for a class of systems representative of large space structures. An analytical solution for the compensator is obtained using the infinite dimensional model of the system. A finite dimensional approximation of the compensator is then derived based upon analysis of the infinite dimensional compensator.

2.0 SYSTEM MODEL

We will consider the following system

$$\mathcal{M}\ddot{w}(t) + A_o w(t) = bu(t) + bn_d(t) \tag{2.1}$$

$$y(t) = [b,\dot{w}(t)] + e(t) \tag{2.2}$$

(2.1) is the inhomogeneous equation of motion of an undamped oscillator and (2.2) is the system measurement equation. $w(t)$ is an element of a separable Hilbert space $\mathcal{H}$ and represents the small displacements of the system (translation and rotational) relative to its equilibrium position. The operator $\mathcal{M}$ contains the mass and inertia properties of the system. $\mathcal{M}$ maps $\mathcal{H}$ to $\mathcal{H}$, is linear bounded, self-adjoint, and positive definite. A_o represents the stiffness of the structure and maps $\mathcal{D}(A_o)$, the domain of A_o which is dense in $\mathcal{H}$, to $\mathcal{H}$. A_o is linear, self-adjoint, closed (or can be closed) and generally unbounded. We assume there exists $\varepsilon>0$ such that

$$[A_o\, w,w] \geq \varepsilon \|w\|^2 , \; w \in \mathcal{D}(A_o)$$

The spatial domain of A_o, Ω, is bounded and, thus, the resolvent of A_o is compact for each λ in the resolvent set of A_o. Since A_o is closed and has a compact resolvent, its eigenvalues are isolated (countable), have finite multiplicities, and have infinity as the only limit point ([5], p.187). In addition, the modes $\{\phi_k,\ k=1,2...\}$ of A_o are orthogonal and form a basis in $\mathcal{H}$.

A single white noise disturbance $n_d(t)$ is present with a spatial force distribution defined by $b \in \mathcal{H}$. The support of b is assumed to be small compared to

the measure of Ω, so that $bn_d(t)$ represents a physical realization of a point disturbance. A single control $u(t)$ and rate sensor $y(t)$ are colocated at the sight of the disturbance. The measurement error $e(t)$ is a white noise process uncorrelated with $n_d(t)$. $u(t)$, $n_d(t)$, $y(t)$, and $e(t)$ are all elements of R for each $t \in [0,\infty)$. Note also that the control, measurement, and disturbance are all compact, since they are each finite in number.

In order to utilize semigroup theory, (2.1) and (2.2) will be put in first order form. Let $x_1(t) = w(t)$ and $x_2(t) = \dot{w}(t)$ then, the system (2.1) and (2.2) can be written as

$$\dot{x}(t) = Ax(t) + Bu(t) + Fn_d(t) \tag{2.3}$$

$$y(t) = Cx(t) + e(t) \tag{2.4}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}A_o & 0 \end{bmatrix}$$

$$B = F = \begin{bmatrix} 0 \\ M^{-1}b \end{bmatrix} \qquad C = [0,\ (M^{-1}b)^*]$$

Note that $\mathcal{D}(A) = \mathcal{D}(A_0) \times V$, where $V = \{w \mid A_o^{1/2} w \in H\}$. Note also that A will have a compact resolvent since A_o does.

As a function space, it is natural to use the energy space $E = V \times H$ defined by the inner product

$$[x,z]_E = [A_o^{1/2} x_1,\ A_o^{1/2} z_1] + [Mx_2,\ z_2],\quad x,z \in E \tag{2.5}$$

It is easy to verify that $A = -A^*$ under the inner product (2.5). Hence, by [6, Corollary 4.3.1], A generates a norm preserving, strongly continuous group, $T(t)$

The semigroup can be represented using the modes of $M^{-1}A_o$. The modes ϕ_k are orthogonal under the inner product $[M\cdot,\cdot]$ and are complete in H. Using the ϕ_k's, a set of basis vectors in E can be defined by $\phi_{1k} = [\phi_k,0]^T$, $\phi_{2k} = [0,\phi_k]^T$, $k=1,2,\ldots$ The set $\{\phi_{1k}, \phi_{2k}, k=1,2,\ldots\}$ is complete in E and orthogonal under $[\cdot,\cdot]_E$. Thus, any $x \in E$ can be written as

$$x = \sum_{k=1}^{\infty} x_{1k}\,\phi_{1k} + x_{2k}\,\phi_{2k}$$

where $[\phi_{1k},x]_E = \omega_k^2\, x_{1k}$, $[\phi_{2k},x]_E = x_{2k}$.

A representation for the semigroup, can be obtained from the homogeneous solution to (2.3). Writing (2.3) as

$$\dot{x} - Ax = \sum_{k=1}^{\infty} (\dot{x}_{1k} - x_{2k})\,\phi_{1k} + (\dot{x}_{2k} + \omega_k^2\, x_{1k})\,\phi_{2k} = 0$$

we see that x_{1k} and x_{2k} solve

$$\begin{bmatrix} \dot{x}_{1k} \\ \dot{x}_{2k} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_k^2 & 0 \end{bmatrix} \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix}$$

for each k. Thus the semigroup can be expressed as

$$\begin{aligned} x(t) &= T(t)\, x(0) \\ &= \sum_{k=1}^{\infty} \left\{ \left[x_{1k}(0)\, \cos\omega_k t + \frac{x_{2k}(0)}{\omega_k} \sin\omega_k t \right] \phi_{1k} \right. \\ &\qquad \left. + \left[-\omega_k\, x_{1k}(0)\, \sin\omega_k t + x_{2k}(0)\, \cos\omega_k t \right] \phi_{2k} \right\} \end{aligned} \tag{2.6}$$

3.0 STOCHASTIC CONTROL PROBLEM

Given the system (2.3) and (2.4), we want to find a control $u(t)$ to minimize

$$J[u] = \lim_{T\to\infty} \frac{1}{T} E \left\{ \int_0^T [\gamma^2\, |B^* x(t)|^2 + |u(t)|^2]\, dt \right\} \tag{3.1}$$

where $\gamma>0$ and $E\{\cdot\}$ denotes expected value. Let the noise covariances be given by $E\, n_d(t)\, n_d^*(t) = \sigma_d^2\, I$ and $E\, e(t)\, e^*(t) = \sigma_s^2$. For (A,B) controllable*, the minimum of (3.1) is attained for

$$u_o(t) = -B^* P_c\, \hat{x}(t) \tag{3.2}$$

where $\hat{x}(t)$ satisfies

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_o(t) + P_f B^* [y(t) + B\hat{x}(t)], \qquad \hat{x}(0) = 0 \tag{3.3}$$

and P_c and P_f satisfy, respectively,

$$\begin{aligned} &[P_c x, Ax] + [Ax, P_c x] + \gamma^2\, [B^* x, B^* x] \\ &\quad - [B^* P_c x, B^* P_c x] = 0, \qquad x \in \mathcal{D}(A) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &[P_f x, A^* x] + [A^* x, P_f x] + \sigma_d^2\, [B^* x, B^* x] \\ &\quad - \frac{1}{\sigma_s^2}\, [B^* P_f x, B^* P_f x] = 0, \qquad x \in \mathcal{D}(A^*) \end{aligned} \tag{3.5}$$

---

* Approximately controllable, see [6, Theorem 4.9.2].

It is easily verified by direct substitution that under the energy inner product

$$P_c = \gamma I \quad \text{and} \quad P_f = \frac{\sigma_s}{\sigma_d} I$$

thus (3.2) and (3.3) become

$$u_o(t) = -\gamma B^* \hat{x}(t) \tag{3.6}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_o(t) + \frac{\sigma_s}{\sigma_d} B[y(t) - B^*\hat{x}(t)], \quad \hat{x}(0) = 0 \tag{3.7}$$

The corresponding minimal cost is

$$J[u_o] = \gamma\left(\frac{\sigma_s}{\sigma_d}\right)\|b\|^2 \left(\gamma + \frac{\sigma_s}{\sigma_d}\right)$$

<u>Remark 3.1</u>. The above solution requires that the steady-state Riccati equations (3.4) and (3.5) have unique positive definite solutions and that $\hat{x}(t)$ be asymptotically stationary. Regarding (3.4) and (3.5), since we have analytical solutions, the only concern is uniqueness and this is guarenteed by (A,B) controllability [8]. The asymptotic stationarity of $\hat{x}(t)$ follows easily through application of [6, Theorem 6.7.1].

<u>Remark 3.2</u>. For more general problems, where for example we replace B in (3.1) with another operator R, which is Hilbert-Schmidt, exponential stabilizability of the system is sufficient to guarantee solutions to the infinite time, stochastic regulator (ITSR) problem [6, Theorem 6.9.1]. However, when compact controls are employed, as is the case here, the system cannot be given a uniform exponential decay rate* unless the open loop system is already uniformly exponentially stable. Note that, here, the open loop system (2.3) is unitary, i.e., $\|T(t)\| = 1$. Hence, the closed loop system can only be strongly stable**.

<u>Remark 3.2</u>. The existence of solutions to the ITSR problem for systems that are only strongly stabilizable is largely an open problem. Recent results concerning strong stability and the steady-state Riccati equation are given in [8], but the sufficient conditions for existence are quite strong, requiring in particular $\|R^*x\| \leq M_1 \|B^*x\|$ for the control problem and $\|F^*x\| \leq M_2\|Cx\|$ for the filtering problem, where M_1 and M_2 are constants.*** One possible physical interpretation of these requirements is that we only attach a performance penalty to those points where we have located actuators and that we locate a sensor at each disturbance source.

* A semigroup T(t) is uniformly exponentially stable if $\exists\ \omega>0,\ M\geq 1 \ni \|T(t)\| \leq Me^{-\omega t},\ t\geq 0$.

** A semigroup is strongly stable if $\|T(t)x\| \to 0$ as $t\to\infty$, $x\in E$.

***R is the state weighting operator, F is the disturbance input operator, and C is the observation operator.

4.0 OPTIMAL COMPENSATOR

Let T(t) be the C_o semigroup generated by A. Then, the solution to (3.7) is given by

$$\hat{x}(t) = \int_o^t T(t-\tau) \left\{ Bu(\tau) + \frac{\sigma_s}{\sigma_d} B[y(\tau) - B^* \hat{x}(\tau)] \right\} d\tau \qquad (4.1)$$

Using (3.6), we can rewrite (4.1) as

$$u(t) = -\int_o^t B^* T(t-\tau) B \left\{ \gamma u(\tau) + \frac{\sigma_s}{\sigma_d} [\gamma y(\tau) - \overline{B^* \hat{x}(\tau)}] \right\} d\tau$$

$$= -\int_o^t B^* T(t-\tau) B \left\{ \left(\gamma + \frac{\sigma_s}{\sigma_d}\right) u(\tau) + \frac{\gamma\sigma_s}{\sigma_d} y(\tau) \right\} d\tau$$

Therefore

$$u(t) + k_u \int_o^t B^* T(t-\tau) Bu(\tau) d\tau = -k_y \int_o^t B^* T(t-\tau) By(\tau) d\tau \qquad (4.2)$$

where $k_u = \gamma + \sigma_s/\sigma_d$, $k_y = \gamma\sigma_s/\sigma_d$. Observe that (4.2) is a Volterra equation. Define the Volterra operator L as

$$Lf = h \; ; \; \int_o^t B^* T(t-\tau) Bf(\tau) d\tau = h(t)$$

Then (4.2) can be written as

$$u + k_u Lu = -k_y Ly \qquad (4.3)$$

Solving for u we obtain the optimal compensator in abstract form

$$\begin{aligned} u_c &= -k_y [1 + k_u L]^{-1} Ly \\ &= -k_y [1 + K] Ly \end{aligned} \qquad (4.4)$$

where K is also a Volterra operator of the form

$$Kf = h \; ; \; \int_o^t K(t,\tau) f(\tau) d\tau = h$$

From [6, p.102-103], K(t,τ) can be computed iteratively and is given by

$$K(t,\tau) = \sum_{n=1}^{\infty} (-1)^{n+1} g_n(t,\tau) \qquad (4.5)$$

where

$$g_n(t,\tau) = g_n(t-\tau) = \int_\tau^t g(t-\sigma) g_{n-1}(\sigma-\tau) d\sigma \qquad (4.6)$$

$$g_1(t-\tau) \;=\; g(t-\tau) \;=\; B^* \, T(t-\tau) \, B \tag{4.7}$$

Noting that

$$\mathcal{L}^n f \;=\; \int_o^t g_n(t-\tau) \, f(\tau) \, d\tau$$

(4.4) can also be written as

$$\begin{aligned} u_c &= - k_y \left\{ 1 - k_u \;+ k_u^2 \, \mathcal{L}^2 - k_u^3 \, \mathcal{L}^3 + \ldots \right\} \mathcal{L} \, y \\ &= - k_y \left\{ \sum_{n=1}^{\infty} (-k_u)^{n-1} \, \mathcal{L}^n \right\} y \end{aligned} \tag{4.8}$$

Since $B^* \, T(t-\tau) \, B$ is uniformly continuous and (4.3) is a Volterra equation, the iteration defined by (4.5)-(4.7) and, hence, the series (4.8) converge for $t \varepsilon [0,T], \tau \leq t, T < \infty$, [6]. The series (4.8) is sometimes called a Neumann series, [9]. Thus, the optimal compensator is of the form

$$u(t) \;=\; - k_y \int_o^t g_o(t-\tau) \, y(\tau) \, d\tau \tag{4.9}$$

where the kernal $g_o(t-\tau)$ is given by the bracketed term in (4.8). Note that (4.8) utilizes the infinite dimension system model via (4.7) which is given in terms of the open loop semigroup operator, $T(t)$, and we have a representation for $T(t)$, (2.6), using the modes of $M^{-1}A_o$. In particular, expanding (4.7) on the orthonormal basis for E, as defined in Section 2, gives

$$g(t-\tau) \;=\; B^* \, T(t-\tau) \, B \;=\; \sum_{k=1}^{\infty} b_k^2 \, \cos\omega_k (t-\tau)$$

where $b_k = [M(M^{-1}b), \, \phi_{2k}] = [b, \, \phi_k]$.

Properties of $g(\sigma)$

i) $g(\sigma) \to 0$ as $\sigma \to \infty$

ii) $g(\sigma)$ is uniformly continuous

iii) $g(\sigma)$ is compact for each $t \geq 0$.

Condition i) follows since $g(\sigma) = B^* \, S(\sigma) \, B$ and $S(\sigma)$ is the C_o semigroup generated by $A - k_u BB^*$ which is strongly stable. Note that $S(\sigma)$ is also a contraction, i.e., $\|S(\sigma)\| \leq 1$. The second condition results from the strong continuity of $S(\sigma)$. For condition iii), $g(\sigma)$ is compact since B is compact and $S(\sigma)$ is a bounded operator.

5.0 SMALL GAIN APPROXIMATION TO OPTIMAL COMPENSATOR

Using the procedure outlined in Section 4.0, four complete interations and a partial fifth iteration were performed. This yielded an approximation for the kernal of the optimal filter given by

$$g_o(\sigma) \tilde{=} \sum_{k=1}^{\infty} b_k^2 \cos\omega_k\sigma \left[1 - \xi_{ck}\sigma + \frac{1}{2}(\xi_{ck}\sigma)^2 - \frac{1}{3!}(\xi_{ck}\sigma)^3 + \frac{1}{4}(\xi_{ck}\sigma)^4\right]$$

$$- \sum_{k=1}^{\infty} \xi_{ck}\, b_k^2 \frac{\sin\omega_k\sigma}{\omega_k}\left[1 - \xi_{ck}\sigma + \frac{1}{2}(\xi_{ck}\sigma)^2 - \frac{1}{3!}(\xi_{ck}\sigma)^3\right]$$

$$- \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \sum_{j=1}^{\infty} \left\{\Omega_{kj} \frac{b_k^2 \sin\omega_k\sigma}{\omega_k}\left[1 - \xi_{ck}\sigma + \frac{1}{2}(\xi_{ck}\sigma)^2\right] + \Omega_{jk} \frac{b_j^2 \sin\omega_j\sigma}{\omega_j}\right.$$

$$\left.\cdot \left[1 - \xi_{cj}\sigma + \frac{1}{2}(\xi_{ck}\sigma)^2\right]\right\} + k_u^2\, 0_4(k_u ; \frac{1}{\omega_k} ; \sigma) \qquad 5.1)$$

where

$$\xi_{ck} = k_u\, b_k^2/2\ , \quad \Omega_{kj} = \omega_k^2\, b_j^2/(\omega_k^2 - \omega_j^2)\ , \qquad \sigma = t - \tau$$

The remainder term $k_u^2\, 0_4(k_u ; \frac{1}{\omega_k} ; \sigma)$ is given in the appendix and is composed of terms of second order or greater in k_u.

It is apparent that the terms in the square brackets in (5.1) are the first few terms of the Taylor series expansion for exp $(-\xi_{ck}t)$. Thus, the limiting form of the kernel is

$$g_o(\sigma) = \hat{g}_o(\sigma) + k_u^2\, 0(k_u; \frac{1}{\omega_k} ; \sigma) \qquad (5.2)$$

where

$$\hat{g}_o(\sigma) = \sum_{k=1}^{\infty} g_{1k}(\sigma) - k_u \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \sum_{j=1}^{\infty} g_{2kj}(\sigma) \qquad (5.3)$$

$$g_{1k}(\sigma) = b_k^2\, e^{-\xi_{ck}\sigma}\left[\cos\omega_k\sigma - \xi_{ck}\frac{\sin\omega_k\sigma}{\omega_k}\right]$$

$$g_{2kj}(\sigma) = \Omega_{kj}\, b_k^2\, e^{-\xi_{ck}\sigma}\left(\frac{\sin\omega_k\sigma}{\omega_k}\right) + \Omega_{jk}\, b_j^2\, e^{-\xi_{cj}\sigma}\left(\frac{\sin\omega_j\sigma}{\omega_j}\right) \qquad (5.5)$$

Note that the component filters $g_{1k}(\sigma)$ have second order dynamics while the filters $g_{2kj}(\sigma)$ have fourth order dynamics. Also, all the parameters in $g_{1k}(\sigma)$ are

related to the kth open loop mode, while $g_{2kj}(\sigma)$ is related to the cross-coupling between the kth and jth modes. Due to the presence of the factors $1/\omega_k$ and $1/\omega_j$ in $g_{2kj}(\sigma)$, the cross-coupling terms are attenuated relative to the "diagonal" terms $g_{1k}(t-\tau)$ as ω_k and ω_j become large. Thus, at high frequencies the dominant filter dynamics are due to the diagonal terms $g_{1k}(\sigma)$. Since the iteration converges and since $g_o(\sigma)$ and $\hat{g}_o(\sigma)$ are continuous in σ, $k_u^2\, 0(k_u;1/\omega_k;\sigma)$ is continuous in σ. Also $k_u^2|0(k_u;1/\omega_k;\sigma)| \to 0$ as $k_u \to 0$ for each $\sigma\varepsilon[0,T]$. Therefore, for any $\varepsilon > 0$, we can find k_u sufficiently small such that

$$\max_{0\le t\le T} \left|k_u^2\ 0(k_u;\frac{1}{\omega_k};\sigma)\right| < \varepsilon$$

The argument $1/\omega_k$ is used to indicate that $|0(k_u;1/\omega_k;\sigma)| \to 0$ as $\omega_k \to \infty$. Therefore, assuming k_u is small, the approximate optimal control is given by

$$\hat{u}(t) = -k_y \int_o^t \hat{g}_o(t-\tau)\ y(\tau)\ d\tau$$

$$= -k_y \sum_{k=1}^{\infty} \int_o^t g_{1k}(t-\tau)\ y(\tau)\ d\tau + k_y\ k_u \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \sum_{j=1}^{\infty} \int_o^t g_{2kj}(t-\tau)\, y(\tau) d\tau \quad (5.6)$$

Since k_y and k_u are both small, as crude first approximation, the double summation in (5.6) could be neglected. In this case, the filter takes a particularly simple form, namely,

$$\hat{u}(t) = \sum_{k=1}^{\infty} \hat{u}_k(t) \quad (5.7)$$

where

$$\hat{u}_k(t) = -k_y \int_o^t g_{1k}(t-\tau)\ y(\tau)\ d\tau$$

Here, the basic structure of the compensator is an infinite bank of second order filters, each operating on the sensor output and the control is simply the sum of their outputs. The component filters $g_{1k}(t-\tau)$ in this case have a rather special property. Letting $G_{1k}(s)$ denote the Laplace Transform of $g_{1k}(t-\tau)$, we have

$$G_{1k}(s) = \frac{b_k^2 s}{s^2 + 2\xi_{ck} s + \xi_{ck}^2 + \omega_k^2}$$

Such a filter is strictly positive real, [11], [12]. Note that the phase shift of the filter never exceeds $\pm\pi/2$. According to [13], strictly positive real filters have robust stability properties in the sense that they will stabilize any

positive real system (in an input/output sense). The system considered here is positive real since it is an undamped oscillator. Note that any finite sum of positive real filters is also positive real. Therefore, we can truncate the summation (5.7) at any finite number of terms and the compensator will still stabilize the system. It would be expected that the more terms retained in the summation the closer the resulting performance would be to the optimal performance.

Returning to the filter defined by (5.6) another approximation will be obtained. First, consider the Laplace Transform of $k_u\, g_{2k_j}(t)$ given by

$$k_u\, G_{2kj}(s) = k_u \left[\frac{b_k^2\, \Omega_{kj}}{(s+\xi_{ck})^2 + \omega_j^2} + \frac{b_j^2\, \Omega_{jk}}{(s+\xi_{cj})^2 + \omega_j^2} \right]$$

Using the definitions of ξ_{ck}, Ω_{kj}, etc, this becomes

$$k_u\, G_{2kj}(s) = k_u \frac{b_k^2\, b_j^2 \left\{ (\omega_k^2 - \omega_j^2)\, s^2 + 2k_u(\omega_k^2\, b_j^2 - \omega_j^2\, b_k^2) + k_u^2(\omega_k^2\, b_j^4 - \omega_j^2\, b_j^4)/4 \right\}}{(\omega_k^2 - \omega_j^2)\left[\left(s + \frac{k_u b_k^2}{2}\right)^2 + \omega_k^2\right]\left[\left(s + \frac{k_u b_j^2}{2}\right)^2 + \omega_j^2\right]}$$

Dropping all terms second order or greater in k_u gives

$$k_u G_{2kj}(s) \doteq k_u \frac{b_k^2\, b_j^2\, s^2}{(s^2 + k_u b_k^2 s + \omega_k^2)\ (s^2 + k_u b_j^2 s + \omega_j^2)} \doteq k_u\, G_{1k}(s)\, G_{1j}(s) \qquad (5.8)$$

where $G_{1k}(s)$ is the Laplace Transform of $g_{1_k}(t)$ (neglecting ξ_{ck}^2, ξ_{cs}^2 in the denominator). Thus the Laplace Transform of the compensator is

$$G_o(s) = -k_y \sum_{k=1}^{\infty} \frac{b_k^2 s}{s^2 + k_u b_k^2 s + \omega_k^2} + k_u k_y \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{j=1}^{\infty} \frac{b_k^2\, b_j^2\, s^2}{(s^2 + k_u b_k^2 + \omega_k^2)\ (s^2 + k_u b_j^2 s + \omega_j^2)} \qquad (5.9)$$

In the time domain, the approximation to the optimal control is now given by

$$\hat{u}(t) = -k_y \sum_{k=1}^{\infty} \int_o^t g_{1k}(t-\tau)\ y(\tau)\ d\tau + k_y k_u \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{j=1}^{\infty} \int_o^t \left\{ \int_\tau^t g_{1k}(t-\sigma)\ g_{1j}(\sigma-\tau) d\sigma \right\} y(\tau) d\tau \qquad (5.10)$$

This representation is interesting in that the basic building blocks of the filter are the positive real component filter $g_{1k}(t-\tau)$. Note, however, that the convolutions

$$g_{3kj}(t,\tau) = \int_{\tau}^{t} g_{1k}(t-\sigma)\ g_{1j}(\sigma-\tau)\ d\sigma$$

or, equivalently, the products

$$G_{3kj}(s) = G_{1k}(s)\ G_{1j}(s)$$

are not positive real. However, since the diagonal terms dominate at high frequencies, the high frequency component filters approach a positive real form. Since we would expect a close (finite dimensional) approximation to the optimal filter to stabilize the system, it is interesting that the limiting structure is positive real, i.e., a structure that is compatible with truncation.

6.0 CONCLUSIONS

It has been shown that the optimal compensator can be represented in terms of a kernal of an integral operator. The results demonstrate that the integral representation provides considerable insight into the structure of the compensator. This is due to the compensator being compact and, hence, something that is inherently approximatible by a finite dimensional operator, i.e., the compensator parameters go to zero as $k \to \infty$. For the system considered here, the optimal filter can be constructed of basic building blocks which are second order positive real filters. The first (low gain) approximation is a diagonal array of these component filters and there is a one to one correspondence with the open loop modes. The second (low gain) approximation adds in the cross-coupling effects in the form of fourth order filters which are convolutions between the various component filters taken two at a time. At high frequencies, the dynamics of the compensator approach the diagonal form of the first approximation which should have advantages when the filter dynamics are truncated in order to obtain a finite dimensional approximation for the compensator. The results also seem to suggest that higher gains will require increasingly higher order convolutions of the component filters.

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APPENDIX

The following defines $O_4(k_u;t)$, the remainder term for the optimal compensator, after four complete iterations and part of a fifth iteration:

$$O_4(k_u;\frac{1}{\omega_k};t) = \left\{\sum_{k=1}^{\infty}\frac{b_k^6 t\sin\omega_k t}{8\omega_k}\left[1-\xi_{ck}t+\frac{1}{2}(\xi_{ck}t)^2\right]\right.$$

$$+\sum_{k\neq j}\sum\left[\frac{\Omega_{kj}\ b_k^2\sin\omega_k t}{\omega_k}\left(c_1(k,j)+\frac{tb_k^4}{8}(1-\xi_{ck}t)\right)+\frac{\Omega_{jk}b_j^2\sin\omega_j t}{\omega_j}\right.$$

$$\cdot\left.\left(c_1(j,k)+\frac{tb_j^4}{8}(1-\xi_{ck}t)\right)\right]+\sum_{k\neq j}\sum\ b_k^2b_j^2\ [(c_2(k,j)+k_uc_3(k,j)t)\ \cos\omega_k t$$

$$+(c_2(j,k)+k_uc_3(j,k)t)\ \cos\omega_j t]-\sum_{k\neq j\neq i}\sum\sum\ b_k^2b_j^2b_i^2\ [(c_4(k,j,i)+k_uc_5(k,j,i)t)$$

$$\left.\cdot\cos\omega_k t+(c_4(j,k,i)+k_uc_5(j,k,i)t)\ \cos\omega_j t+(c_4(i,j,k)+k_uc_5(i,j,k))\ \cos\omega_i t]\right\}$$

$$+k_u\left\{\sum_{k=1}^{\infty}\frac{b_k^8 t\cos\omega_k t}{8\omega_k}-\frac{b_k^2\sin\omega_k t}{12\omega_k^2}-\sum_{k\neq j\neq i}\sum\sum\ b_k^2b_j^2b_i^2\ [c_6(k,j,i)\ \sin\omega_k t\right.$$

$$+c_6(j,k,i)\ \sin\omega_j t+c_6(i,j,k)\sin\omega_i t]-\sum_{k\neq j\neq i\neq \ell}\sum\sum\sum b_k^2b_j^2b_i^2b_\ell^2\ [c_7(k,j,i,\ell)\ \sin\omega_k t$$

$$\left.+c_7(j,k,i,\ell)\ \sin\omega_j t+c_7(i,j,k,\ell)\ \sin\omega_i t+c_7(\ell,j,i,k)\ \sin\omega_\ell t]\right\}$$

where

$$c_1(k,j) = \frac{2}{(\omega_k^2-\omega_j^2)^2}\left[b_j^4\omega_k^2+\frac{\omega_j^4b_k^4}{\omega_k^2}-\frac{3b_j^2b_k^2\omega_j^2}{\omega_k^2}\right]-\frac{2}{\omega_k^2-\omega_j^2}\left[\frac{(\omega_k^2-2\omega_j^2)b_k^4}{\omega_k^2}+\frac{3b_j^2}{4}\right]+\frac{3b_k^4}{4\omega_k^3}$$

$$c_2(k,j) = \frac{3}{2(\omega_k^2-\omega_j^2)^2}\ (\omega_j^2b_k^2-\omega_k^2b_j^2)$$

$$c_3(k,j) = \frac{1}{(\omega_k^2-\omega_j^2)^2}\left[2(\omega_k^2-2\omega_j^2)\ b_k^4+\frac{\omega_k^2b_k^2b_j^2}{2}\right]-\frac{3b_k^4}{\omega_k^2-\omega_j^2}$$

$$c_4(k,j) = \frac{\omega_k^2b_k^2}{(\omega_j^2-\omega_k^2)(\omega_i^2-\omega_k^2)}\ ,\quad c_5(k,j,i) = \frac{\omega_k^2b_k^2}{(\omega_i^2-\omega_k^2)(\omega_j^2-\omega_i^2)}$$

$$c_6(k,j,i) = \frac{1}{(\omega_j^2-\omega_k^2)(\omega_i^2-\omega_k^2)}\left[4\omega_k^3 b_i^2 + \frac{2(\omega_k^3(\omega_i^2+\omega_j^2) - 2\omega_k\omega_j^2\omega_i^2)b_k^2}{(\omega_j^2-\omega_k^2)} + \omega_k b_k^2\right]$$

$$c_7(k,j,i,\ell) = \frac{\omega_k^3}{(\omega_i^2-\omega_k^2)(\omega_j^2-\omega_k^2)(\omega_\ell^2-\omega_k^2)}$$

$$\xi_{ck} = k_u b_k^2/2 \quad , \quad \Omega_{kj} = \omega_k^2 b_j^2/\omega_k^2-\omega_j^2$$

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